

Lecture 5 : Exponential Growth and Decay

Many quantities grow or decay at a rate proportional to their size. Such quantities give us an equation of the form

$$\frac{dy}{dt} = ky.$$

called a **differential equation** because it gives a relationship between a function and one or more of its derivatives. If $k < 0$, the above equation is called **the law of natural decay** and if $k > 0$, the equation is called **the law of natural growth**.

A solution to a differential equation is a function y which satisfies the equation. It is not difficult to see that $y(t) = e^{kt}$ is one solution to the above differential equation. However as with antiderivatives, the above differential equation has many solutions. We see that any function of the form

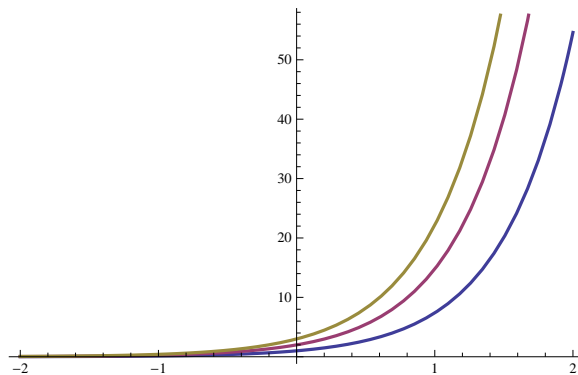
$$y(t) = Ce^{kt}$$

is a solution for any constant C . We will prove later that every solution to the differential equation above has the form $y(t) = Ce^{kt}$. Setting $t = 0$, we get

The **only solutions to the differential equation** $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

Here is a picture of three solutions to the differential equation $dy/dt = 2y$, each with a different value $y(0)$.



Population Growth Let P be the size of a population at time t . The law of natural growth is a good model for population growth (up to a certain point):

$$\frac{dP}{dt} = kP \quad \text{and} \quad P(t) = P(0)e^{kt}$$

Note that the relative growth rate, $\frac{dP}{dt}/P = k$ is constant.

Example The population of Mathland at the end the year 2000 was 500. The population increases (continuously or steadily) by approximately 10% per year. What is the function $P(t)$, the size of the population after t years, using the exponential model above?

What differential equation does the function $P(t)$ satisfy?

What is the value of k ?

What is $P(0)$?

Give a formula for $P(t)$

What will the population be in the year 2050?

Example The population of Calculand was 700 in the year 2000 and was 3000 in the year 2010. Using the exponential model for population growth, find an estimate for the population of Calculand in 2015.

$P(0) = ?$

$P(10) = ?$

Find the value of k .

Find the formula for $P(t)$ and use it to find $P(15)$.

Radioactive Decay Radioactive substances decay at a rate proportional to their mass.

$$\boxed{\frac{dm}{dt} = km \quad \text{and} \quad m(t) = m_0 e^{kt},}$$

where $m(t)$ denotes the mass of the substance at time t and m_0 denotes the mass of the substance at time $t = 0$. **The half-life of a radioactive substance is the time required for half of the quantity to decay.**

Example the half-life of Carbon-14 is approximately $t_{1/2} = 5,730$ years (there is some variety in this depending on variables such as location). When a plant or animal dies, it stops taking in Carbon and the carbon it contains starts to decay.

Example A bowl made of oak has about 40% of the carbon-14 that a similar quantity of living oak has today. Estimate the age of the bowl.

Use the half-life to find the value of k

Find a formula for $m(t)$ = amount carbon-14 after t years.

Solve for t using the fact that there is 40% of the original amount now present.

Interest If we invest $\$A_0$ in an account paying $r \times 100$ % interest per annum and the interest is compounded continuously, the amount in the account after t years is given by

$$A(t) = A_0 e^{rt}.$$

Example If I invest \$1000 for 5 years at a 4% interest rate with the interest compounded continuously, how much will be in my account at the end of the 5 years?

We are given that $A_0 = 1000$ and $r = 0.04$. Because the interest is compounded continuously, we have

$$A(t) = A_0 e^{0.04t} = 1000e^{0.04t}$$

and

$$A(5) = 1000e^{0.04(5)} = \$1221.4.$$

How long before there is \$2000 in the account?

Compound Interest If I invest $\$A_0$ in an account with an interest rate of $r \times 100\%$ per annum, the amount in the bank account after t years depends on the number of times the interest is compounded per year.

In the chart below

$A_0 = A(0)$ is the initial amount invested at time $t = 0$.

$A(t)$ is the amount in the account after t years.

n = the number of times the interest is compounded per year.

We have

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

| Amt. after t years | $A(0)$ | $A(1)$ | $A(2)$ | ... | $A(t)$ |
|--|------------------|---|---|-----|---|
| $n = 1$ | A_0 | $A_0(1 + r)$ | $A_0(1 + r)^2$ | ... | $A_0(1 + r)^t$ |
| $n = 2$ | A_0 | $A_0(1 + \frac{r}{2})^2$ | $A_0(1 + \frac{r}{2})^4$ | ... | $A_0(1 + \frac{r}{2})^{2t}$ |
| $n = 12$ | A_0 | $A_0(1 + \frac{r}{12})^{12}$ | $A_0(1 + \frac{r}{12})^{24}$ | ... | $A_0(1 + \frac{r}{12})^{12t}$ |
| | \vdots | \vdots | \vdots | | \vdots |
| n | A_0 | $A_0(1 + \frac{r}{n})^n$ | $A_0(1 + \frac{r}{n})^{2n}$ | ... | $A_0(1 + \frac{r}{n})^{nt}$ |
| | \vdots | \vdots | \vdots | | \vdots |
| $n \rightarrow \infty$ (compounded continuously) | A_0 $= A_0$ | $\lim_{n \rightarrow \infty} A_0(1 + \frac{r}{n})^n$ $= A_0 e^r$ | $\lim_{n \rightarrow \infty} A_0(1 + \frac{r}{n})^{2n}$ $= A_0 e^{2r}$ | ... | $\lim_{n \rightarrow \infty} A_0(1 + \frac{r}{n})^{nt}$ $= A_0 e^{rt}$ |

Note that

$$\lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{\frac{n}{r} \cdot \frac{r}{t} \cdot t} = \lim_{n \rightarrow \infty} A_0 \left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{\frac{r}{t} \cdot t} = A_0 \left(\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{\frac{r}{t} \cdot t} = A_0 e^{rt}.$$

Example If I borrow $\$50,000$ at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?

Extras and Solutions

Example A baby was 20 in long when born and was 3 ft. tall at age 2 years. If this person grows exponentially, what will her height be when she is 20 years old? Is the exponential growth model appropriate here?

Example A bowl made of oak has about 40% of the carbon-14 that a similar quantity of living oak has today. Estimate the age of the bowl.

$$m(t) = m(0)e^{kt}$$

To find k , we use the half-life of Carbon-14:

$$\frac{m(5,730)}{m(0)} = \frac{1}{2} = \frac{m(0)e^{5730k}}{m(0)} = e^{5730k}.$$

Applying the natural logarithm, we get

$$\ln\left(\frac{1}{2}\right) = \ln(e^{5730k}) = 5730k$$

giving us that

$$k = \frac{\ln(\frac{1}{2})}{5730}.$$

To find the age we solve for the time t when the Carbon-14 had decayed to 40% of its original value. We solve for t is

$$\frac{m(t)}{m(0)} = \frac{m(0)e^{kt}}{m(0)} = .4$$

that is

$$e^{kt} = .4 \quad \text{or} \quad \ln(e^{kt}) = \ln(.4) \quad \text{or} \quad kt = \ln(.4)$$

This gives

$$t = \frac{\ln(.4)}{\frac{\ln(\frac{1}{2})}{5730}} \approx 7575 \text{ years.}$$

The formula used for reference by scientists

$$t = \frac{\ln(M/M_0)}{\ln(1/2)}t_{1/2}.$$

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$$A(t) = A_0 e^{0.04t} = 1000e^{0.04t}$$

and

$$A(5) = 1000e^{0.04(5)} = \$1221.4.$$

How long before there is \$2000 in the account? If

$$A(t) = 2000$$

then

$$2000 = 1000e^{0.04t} \quad \text{or} \quad 2 = e^{0.04t}$$

Applying the natural logarithm, we get

$$\ln 2 = 0.04t \quad \text{or} \quad t = \frac{\ln 2}{0.04} = 17.33.$$

Example If I borrow \$50,000 at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?

$$A_0 = 50,000$$

Because the interest is compounded quarterly,

$$A(t) = A_0 \left(1 + \frac{0.1}{4}\right)^{4t}$$

and

$$A(5) = 50,000(1 + 0.025)^{20} = 81,930.82$$

(The 82 cents is the bank's profit :))